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New records in Stanley–Wilf limits

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Abstract

We provide examples for patterns q for which the value of the Stanley–Wilf limit $L(q)$ is smaller or larger than any previously known values. The exact values of $L(q)$ are computed. Generalizations are given.

1. Introduction

Let $S_n(q)$ be the number of permutations of length n (or, in what follows, n-permutations) that avoid the pattern q. For a brief introduction to the area of pattern avoidance, see [5]; for a more detailed introduction, see [6]. A recent spectacular result of Marcus and Tardos [9] shows that for any pattern q, there exists a constant c_q such that $S_n(q) < c_q^n$ holds for all n. As pointed out by Arratia in [2], this is equivalent to the statement that $L(q) = \lim_{n\to\infty} \sqrt[n]{S_n(q)}$ exists; in fact the sequence $\sqrt[n]{S_n(q)}$ is always monotone increasing and convergent. Let us call the sequence $\sqrt[n]{S_n(q)}$ a *Stanley–Wilf* sequence, and $L(q)$ a *Stanley–Wilf limit*. It is a natural and intriguing question to ask what the limit $L(q)$ of a Stanley–Wilf sequence can be, for various patterns q.

2. Relevant recent results and list of our new results

For more than a decade, in all cases when $L(q)$ was known, it was known to be an integer. However, it was shown in [7] that $L(q)$ is not even always rational. As far as the size of $L(q)$ goes, it was conjectured by Arratia that $L(q) \le (k-1)^2$ where k is the length of q, but this has recently been disproved in [1], where M. Albert et al. showed that $a = L(1324) \ge 9.35$. Using

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techniques from [7], it is straightforward to extend this to the inequality $L(13245678\cdots k) = (\sqrt{a}+(k-4))^2 \geq (\sqrt{9.35}+(k-4))^2$. Before the present paper, this has been the highest known example of $L(q)$, for a pattern q of length k.

As far as *the lowest* known example of $L(q)$ goes, an absolute lower bound $L(q) \geq (k-1)^2/e^3$ is known by an argument of Valtr [8] for any patterns q of length k. Before the present paper the lowest known example was obtained in [7], where it was shown that $(31245678\cdots k) = (\sqrt{8}+(k-4))^2$. Note that $L(q) \geq k-1$ is fairly easy to prove for any pattern $q \in S_k$, and for small values of k, this provides a better lower bound than the inequality $L(q) \geq (k-1)^2/e^3$.

In this paper, we improve the examples on both the lowest and the highest values of $L(q)$. For the lowest value, we show that $L(3124675) = 32$, which improves the above-mentioned lowest value of $L(3124567) = ((\sqrt{8}+3)^2 = 17+6\sqrt{8}) = 33.97$. For the highest value, we show that $L(1324657) = 4a$, where $a = L(1324) \geq 9.35$. This beats the previously known higher value of $L(1324567) = (\sqrt{a}+3)^2 = a+9+6\sqrt{a}$ as can be easily seen by comparing the square roots of the two limits.

Both results will be generalized to longer patterns. The proofs will be based on a significantly enhanced version of the method of [7], but will be written in a self-contained manner.

3. An example with a low Stanley–Wilf limit

3.1. The upper bound

In this section we show that $L(3124675) = 32$, which is the first time any pattern of length k is shown to have a Stanley–Wilf limit that is less than $(\sqrt{8}+(k-4))^2$. We start by showing that this limit is *at most* 32^n.

Lemma 3.1. *For all positive integers n,*

$$S_n(3124675) \leq 32^n.$$

Proof. Let p be an n-permutation that avoids 3124675. Let us color all entries of p that can play the role of 4 in a 3124-pattern of p red, and let us color all other entries of p blue. (In some very imprecise sense, the reader may think of red entries as "big" and blue entries as "small".) Then the red entries form a 1342-avoiding permutation, because if q was a copy of 1342 consisting of red entries and starting in the entry a, then the copy of 3124 that qualifies a to be a red entry together with the other three entries of q would form a 3124675-pattern. The blue entries of p form a 3124-avoiding permutation, since by definition, no blue entry can ever play the role of 4 in a 3124-pattern.

Now assume p has b red entries. Then there are at most $\binom{n}{b}$ choices for the set of these entries, and at most $\binom{n}{b}$ choices for their positions in p. Then, there are $S_n(1342)$ possibilities for the partial permutation of the red entries, and $S_n(3124)$ possibilities for the partial permutation of the blue entries. Recall [4] that for all n, the inequality $S_n(1342) < 8^n$ holds. Note that 3124 is the *reverse complement* of 1342, and this implies that $S_n(1342) = S_n(3124)$ for all n. Therefore,

$$\begin{aligned} S_n(3124675) &\leq \sum_{b=0}^{n} \binom{n}{b}^2 S_n(1342) S_n(3124) \\ &\leq \sum_{b=0}^{n} \binom{n}{b}^2 8^b 8^{n-b} = 8^n \sum_{b=0}^{n} \binom{n}{b}^2 \end{aligned}$$

$$\leq 8^n \left(\sum_{b=0}^{n} \binom{n}{b} \right)^2 = 32^n.$$

This shows that $L(q) \leq 32$. □

3.2. The lower bound

3.2.1. Background and definitions

Now we are going to prove that the upper bound 32^n is the best possible for this pattern in the sense that $L(3124675) = 32$. To this end, we will need to find a good lower bound for $S_n(3124675)$. Using an enhanced version of our argument in [7], we will show that the method we used to prove the upper bound of Lemma 3.1 did not allow for too much waste, that is, that upper bound is quite close to the truth.

Where *is* the waste in that proof? The waste is that there are some choices for the red entries (that is, their set, their position, or their permutation) that are incompatible with some choices for the blue entries. This is a crucial concept of the upcoming proof, so we will make it more precise.

Definition 3.2. Let n be a positive integer, and let $m \leq n$ be a positive integer. Let U and V be two m-element subsets of $[n]$. Finally, let S be a permutation of the elements of U, and let T be a permutation of the elements of $[n] - U$.

If there exists an n-permutation p such that its blue entries are precisely the elements of U, they are located in positions belonging to V, the (partial) permutation of the blue entries of p is S, and the permutation of the red entries of p is T, then we say that the 4-tuple (S, T, U, V) is *compatible*. Otherwise, we say that the 4-tuple (S, T, U, V) is *incompatible*.

Note that if (S, T, U, V) is *compatible*, then there is exactly one permutation p satisfying all criteria specified by (S, T, U, V).

Example 3.3. If $n = 6$, $m = 2$, then the 4-tuple $(6312, 54, \{1, 2, 3, 6\}, \{1, 2, 3, 4\})$ is compatible, as shown by the permutation 631254.

Example 3.4. If S (the permutation of blue entries) is 312-avoiding, then (S, T, U, V) is always incompatible if $m > 0$. Indeed, the leftmost red entry must be preceded by a 312-pattern of blue entries.

How can we check whether a particular 4-tuple (S, T, U, V) is compatible? We need to construct the only permutation p in which the elements of U form the permutation S, and the elements of V form the permutation T. Then, we must check that the blue entries of p are indeed precisely the elements of U. That is, we must check that the elements of U have the property that they cannot play the role of 4 in a 3124-pattern, and that no other entries of p have that property.

Permutations of a particular kind will be useful throughout this paper. Let N be a positive integer such that $S_n(3124) = S_n(1342) > 7.99^n$ for all $n \geq N$. We know from [4] that such an N exists as $L(1342) = 8$.

Assume now that s is divisible by N. Let us call an s-permutation p *block structured* if we can cut p into s/N blocks of N (consecutive) entries each, so that the entries of any given block B are all smaller than the entries of any block on the left of B, and larger than the entries of any block on the right of B.

If s is not divisible by N, that is, when $s = Nt + r$ for some $r \in [1, N-1]$, then we call S block-structured if its last r entries are its smallest entries, and they are in decreasing order, and its first $s-r$ entries have the block-structured property in the above sense.

Importantly, both 3124 and 1342 are *indecomposable* patterns, that is, they cannot be cut into two parts so that each entry before the cut is larger than each entry after the cut. This implies that if each block of a block-structured permutation p avoids 3124 (resp. 1342), then p itself avoids 3124 (resp. 1342). Therefore, the definition of N implies that there are more than 7.99^{s-r} block-structured s-permutations that avoid 3124, and there are more than 7.99^{s-r} block-structured s-permutations that avoid 1342. Recall that r is the remainder of s modulo N.

We claim that n-permutations in which the partial permutation of the red entries and the partial permutation of the blue entries are both block-structured will provide us with the needed lower bound. Just as in [7], we will achieve this by "imitating" 123-avoiding permutations, which have a simple structure. We need to introduce some standard machinery here. In a permutation p, an entry is called a *left-to-right minimum* if it is smaller than all entries on its left. Entries that are not left-to-right minima are called remaining entries. It is a direct consequence of the definitions that the left–right minima of p form a decreasing subsequence. If p is 123-avoiding, then the remaining entries must form a 123-avoiding sequence too, showing that 123-avoiding permutations are in fact unions of two disjoint decreasing subsequences, the left-to-right minima, and the remaining entries.

Example 3.5. If $p = 35142$, then the left-to-right minima of p are 3 and 1, and the remaining entries of p are 5, 4, and 2.

The following proposition is well known. It relates to a property of 123-avoiding permutations that will be of crucial importance to us.

Proposition 3.6. *Let $1 \leq m \leq n$. Then the number of 123-avoiding n-permutations having exactly m left-to-right minima is*

$$A(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}, \tag{1}$$

a Narayana *number.*

See [10] or [6] for a proof. This result is very significant for us for the following reason. A rough estimate on the number of 123-avoiding n-permutations with m left-to-right minima could go as follows. There are at most $\binom{n}{m}$ possibilities for the set of left-to-right minima, and there are at most $\binom{n}{m}$ possibilities for the positions in which the left-to-right minima are located. As these data determine a 123-avoiding permutation, the number of such permutations with m left-to-right minima is at most $\binom{n}{m}^2$. The above formula shows that roughly $\frac{1}{n}$ of these possibilities will actually be good, that is, they will not violate any constraints, and they will lead to 123-avoiding permutations in which the set and position of left-to-right minima are indeed what they were prescribed to be. The factor $\frac{1}{n}$ is not a significant loss from our point of view, since $\lim_{n\to\infty} \sqrt[n]{1/n} = 1$.

3.2.2. An overview of the construction

Here is an overview of our proof for a lower bound on $S_n(3124675)$. All permutations can be divided into their left-to-right minima and their remaining entries. In a 123-avoiding permutation, both of these sets are arranged in decreasing order. Our construction will take a 123-avoiding permutation, replace the left-to-right minima by 3124-avoiding but 312-containing permutations (the blue entries), and will replace the remaining entries by a 1342-avoiding permutation (the red entries). In order to make sure that the permutation obtained is indeed 3124675-avoiding, we place additional constraints on all three permutations involved. The constraint on the permutation of the blue entries and the red entries is that they are *block-structured*. The constraint on the original 123-avoiding permutation is more complex, and describes how the left-to-right minima and the remaining entries are allowed to interleave. The constraints, in the sense of logarithmic asymptotics, will turn out to be satisfied almost surely, and therefore, will not weaken our bounds on Stanley–Wilf limits.

3.2.3. The details of the construction

Let us start with a 123-avoiding permutation y of length n, and let y' (resp. y'') denote the string of left-to-right minima (resp. remaining entries) of y. Let us choose y so that it satisfies the following additional requirement.

Requirement 3.7. *Each entry $z \in y''$ has the property that if we move z to the left by at most N slots in y'', then z will still be larger than the $2N$ left-to-right minima immediately preceding z in its new position.*

In particular, we require that each entry $z \in y''$ be preceded by at least $2N$ left-to-right minima in y.

For instance, if y has the property that all of its left-to-right minima are smaller than all of its remaining entries, like in $y = 43218765$, and we (unrealistically) assume that $N = 2$, then y satisfies Requirement 3.7.

Now we are going to construct 3124675-avoiding permutations starting from y. Let us replace y' with a block-structured permutation p'_1 in which each block avoids 3124 *but contains 312*, and let us replace y'' with a block-structured permutation p''_1 that avoids 1342. Note that this means no entry moves away more than N positions from its original position in y' or y''. Call the new permutation obtained p.

Note that if y satisfies Requirement 3.7, then each $z \in p''$ is larger than all $x \in p'$ on the right of z. Indeed, as p' is block-structured, each entry of p' is smaller than all entries that precede it by at least $N + 1$ positions in y'. That suffices for proving our claim since z is larger than the $2N$ entries of p' that immediately precede z.

We claim that p avoids 3124675. Indeed, if p contained a copy q of 3124675, then it follows from the block-structured property of both p' and p'', and the fact that each entry of p'' is larger than all entries of p' on its right, that the first i elements of q would have to come from a block of p', and the remaining $7 - i$ elements of q would have to come from a block of p''. However, this is impossible, because $\max(i, 7 - i) \geq 4$, and blocks of p' cannot contain 3124, and blocks of p'' cannot contain 1342.

We point out that we used the original 123-avoiding permutation y to define U and V of the 4-tuple (S, T, U, V), and then we allowed S and T to be any permutations as long as they were block-structured, each block of S avoided 3124 but contained 312, and each block of T avoided 1342. We will now show that all the 4-tuples (S, T, U, V) obtained in this way are compatible, that is, the red entries are indeed the entries of p'', and the blue entries are indeed the entries of p'.

Indeed, Requirement 3.7 assures that in p, each entry z of p'' is still larger than the $2N$ entries of p' immediately preceding z. These $2N$ consecutive entries of p' must contain a full block, and therefore, a 312-pattern Q. Then Qz is a 3124-pattern, proving that z is indeed red. So all entries of p'' are red, and all entries of p' are blue since p' avoids 3124. So compatibility of our 4-tuples is proved.

It remains to show that a sufficient number of 123-avoiding n-permutations satisfy Requirement 3.7. We will do this in two steps. First, let w be a 123-avoiding permutation of length $n - 3N$. Let us add $2N$ to each entry of w, then append the decreasing string $(2N)(2N - 1) \cdots 21$ to the end of w, to get the permutation v, which now has length $n - N$. Finally, move up each left-to-right minimum of v by $2N$ positions *within the string* v' of left-to-right minima of v, to obtain the permutation t. Then the set of left-to-right minima of t and v coincide, and t has the property that each of its remaining entries is larger than the $2N$ left-to-right minima immediately preceding them (since those left-to-right minima all moved up $2N$ positions within their own string). Note that in particular, t starts with a string of at least $2N$ left-to-right minima preceding the leftmost remaining entry since the first $2N$ left-to-right minima of v got pushed to the front of t.

Remember, we are looking for permutations whose remaining entries have the property described in the previous sentence even after moving up N positions in the string of remaining entries. In order to get such a permutation, prepend v with the decreasing string $n(n-1)\cdots(n-N+1)$, to get the n-permutation u, then move each remaining entry of u back by N positions in the string u'' of remaining entries of u. This will make the first N positions of u'' empty; fill those positions with the decreasing string $n(n-1)\cdots(n-N+1)$. The last N remaining entries will "overflow"; put them at the end of the permutation.

The obtained permutation y satisfies Requirement 3.7, since the "old" remaining entries of y moved N positions back in the string of remaining entries, and all the "new" remaining entries, that is, $n, (n-1), \ldots, (n-N+1)$, are larger than all left-to-right minima anyway.

Given y, one can uniquely recover u, and then w. Therefore, the number of 123-avoiding permutations y satisfying Requirement 3.7 is at least as large as the number of 123-avoiding permutations w of length $n - 3N$. This is well known [6] to be the Catalan number $C_{n-3N} = \frac{\binom{2(n-3N)}{n-3N}}{n-3N+1}$.

Now we are in a position to prove the promised lower bound on $S_n(3124675)$.

As we have just seen, there are $\frac{\binom{2(n-3N)}{n-3N}}{n-3N+1}$ choices for the permutation y that avoids 123 and satisfies Requirement 3.7. Once y is known, we need to select the permutations by which we replace y_1 and y_2. If the constant N is sufficiently large, then we have more than 7.99^N choices for each block of y_1, and each block of y_2. (The requirement that each block of y_1 must contain a 312-pattern does not cause a significant loss since it is well known that only $C_N < 4^N$ permutations of length N do not contain a 312-pattern.) Therefore, taking into account that n may be not divisible by N, there are more than 7.99^{n-2N} choices for the pair (y_1, y_2). This proves that

$$S_n(3124675) \geq 7.99^{n-2N} \frac{\binom{2(n-3N)}{n-3N}}{n-3N+1}.$$

Here the constant 7.99 can certainly be replaced by any constant less than 8. As N is a constant, a routine application of Stirling's formula shows that $\sqrt[n]{\frac{\binom{2(n-3N)}{n-3N}}{n-3N+1}} \to 4$ as n goes to infinity. This shows that $L(3124675) \geq 32$, which, together with Lemma 3.1, proves the following theorem.

Theorem 3.8. *The equality* $L(3124675) = 32$ *holds.*

Note that if the original 123-avoiding $n-3N$-permutation w has $m-2N$ left-to-right minima, then y has m left-to-right minima as the first step increased the number of left-to-right minima by $2N$, and the second step did not change their number. By Proposition 3.6, this implies the following.

Proposition 3.9. *Let $m \geq 2N+1$. Then the number of 123-avoiding n-permutations satisfying Requirement 3.7 and having m left-to-right minima is at least*

$$\frac{1}{n-3N}\binom{n-3N}{m-2N}\binom{n-3N}{m-2N-1}.$$

We will need this fact in the next section.

4. Generalizations

The concept of concatenating two permutation patterns often surfaces in the theory of pattern avoidance. In order to facilitate its relevance here, we introduce the following notation.

Definition 4.1. Let $p \in S_a$, and $q \in S_b$, with $p = p_1p_2\dots p_a$ and $q = q_1q_2\dots q_b$. Then the *direct sum* of p and q is the pattern $p \oplus q \in S_{a+b}$ where

$$(p\oplus q)_i = \begin{cases} p_i & \text{if } i \leq a, \\ q_{i-a}+a & \text{if } i > a. \end{cases}$$

In other words, we increase each entry of q by a before placing q after p. For instance, if $p = 132$ and $q = 2431$, then $p \oplus q = 1325764$.

Theorem 3.8 can be generalized in the following way.

Theorem 4.2. *For all patterns q_1 and q_2,*

$$\sqrt{L(q_1\oplus 1\oplus q_2)} = \sqrt{L(q_1\oplus 1)} + \sqrt{1\oplus L(q_2)}.$$

Before we start the proof, note that Theorem 3.8 is the special case of this theorem when $q_1 = 312$ and $q_2 = 231$. We also point out that in [7], we proved this theorem in the very special case of $q_1 = 1$.

Proof. Set $q = q_1 \oplus 1 \oplus q_2$. We first show that $\sqrt{L(q)} \leq \sqrt{L(q_1\oplus 1)} + \sqrt{1\oplus L(q_2)}$. The proof is very similar to that of Lemma 3.1, but the computational part needs a little bit more thought. Let p be a q-avoiding n-permutation, and let us call an entry of p red if it can play the role of the entry $|q_1|$ in q. If an entry is not red, call it blue. Then an argument analogous to that in the proof of Lemma 3.1 shows that

$$\begin{aligned} S_n(q) &\leq \sum_{m=0}^{n}\binom{n}{m}^2 S_n(q_1\oplus 1)S_n(1\oplus q_2) \\ &\leq \sum_{m=0}^{n}\binom{n}{m}^2 L(q_1\oplus 1)^m L(1\oplus q_2)^{n-m} \\ &\leq \left(\sum_{m=0}^{n}\binom{n}{m}\sqrt{L(q_1\oplus 1)^m L(1\oplus q_2)^{n-m}}\right)^2 \\ &= \left(\sqrt{L(q_1\oplus 1)}+\sqrt{L(1\oplus q_2)}\right)^n. \end{aligned}$$

Taking nth roots, this proves our claim that $L(q) \leq L(q_1 \oplus 1) + L(1 \oplus q_2)$.

Now we prove that $\sqrt{L(q)} \geq \sqrt{L(q_1 \oplus 1)} + \sqrt{L(1 \oplus q_2)}$. Use the above definition of red and blue entries, and construct q-avoiding permutations starting from a 123-avoiding permutation y, just as in the proof of Theorem 3.8. Again, new ideas will only be needed at the computational part.

First, note that as $q_1 \oplus 1$ ends in its largest entry and $1 \oplus q_2$ starts in its smallest entry, both $q_1 \oplus 1$ and $1 \oplus q_2$ are indecomposable. This implies that if each block of a block-structured permutation p avoids $q_1 \oplus 1$ or $1 \oplus q_2$, then so does p itself.

In the proof of Theorem 3.8, we chose the permutation of the blue entries, T, so that it avoided 3124, but contained 312. The number of such permutations of length N was $S_n(3124) - S_n(312)$, which is, for large N, roughly $8^N - 4^N$. We were able to neglect the term 4^N since $4 < 8$. In order to be able to do this in the current, more general set-up, we need to show that $L(q_1) < L(q_1 \oplus 1)$. This inequality seems intuitively obvious, but has apparently never been formally put on record. In the special case when q_1 is indecomposable (which covers most, but not all patterns q_1), a stronger result is contained in [7]. We do not want to break the course of the proof of Theorem 4.2 by proving this innocuous claim, so we postpone its proof until Proposition A.1.

Let $\epsilon > 0$, and let us choose N so large that the number of N-permutations containing q' but avoiding $q_1 \oplus 1$ is more than $(L(q_1 \oplus 1) - \epsilon)^N$, and the number of N-permutations avoiding $1 \oplus q_2$ is more than $(L(1 \oplus q_2) - \epsilon)^N$.

We can now construct a q-avoiding n-permutation p in the same way as in Theorem 3.8, with $q_1 \oplus 1$ playing the role of 3124, and $1 \oplus q_2$ playing the role of 1342. As $L(q_1 \oplus 1)$ and $L(1 \oplus q_2)$ may be different, the computation is again more involved.

Let us assume we want to construct a q-avoiding permutation p with m blue entries and $n - m$ red entries. Then Proposition 3.9 shows that there are $\frac{1}{n-3N}\binom{n-3N}{m-2N}\binom{n-3N}{m-2N-1}$ possibilities for the role of y. Therefore, the total number of permutations p that we can construct in this case is more than

$$\frac{1}{n-3N}\binom{n-3N}{m-2N}\binom{n-3N}{m-2N-1}(L(q_1 \oplus 1) - \epsilon)^{m-N+1}(L(1 \oplus q_2) - \epsilon)^{n-m-N+1}.$$

Here the $(-N + 1)$-terms in the exponents are needed as it could be that n is not divisible by N. By routine computation, $\binom{n-3N}{m-2N+1} \geq \frac{1}{n-3N}\binom{n-3n}{m-2N}$. Summing over all m satisfying $N \leq m \leq n - N$, this yields

$$S_n(q) > \frac{1}{(n-3N)^2}\sum_{m=2N+1}^{n}\binom{n-3N}{m-2N}^2(L(q_1 \oplus 1) - \epsilon)^{m-N+1}(L(1 \oplus q_2) - \epsilon)^{n-m-N+1}.$$

As N, $L(q_1 \oplus 1)$ and $L(1 \oplus q_2)$ are constants, this implies that there is a constant C such that

$$Cn^2 \cdot S_n(q) > \sum_{m=2N}^{n-N}\binom{n-3N}{m-2N}^2(L(q_1 \oplus 1) - \epsilon)^{m-2N}(L(1 \oplus q_2) - \epsilon)^{n-N-m},$$

or, setting $i = m - 2N$,

$$Cn^2 \cdot S_n(q) > \sum_{i=0}^{n-3N}\binom{n-3N}{i}^2(L(q_1 \oplus 1) - \epsilon)^{i}(L(1 \oplus q_2) - \epsilon)^{n-3N-i}. \quad (2)$$

Let us now resort to the well-known Cauchy–Schwarz inequality stating that if $a_1, a_2, \ldots, a_d$ are positive real numbers, then

$$\frac{1}{d}(a_1 + a_2 + \cdots + a_d)^2 \leq a_1^2 + a_2^2 + \cdots + a_d^2. \tag{3}$$

The right-hand side of (2) can be viewed as the sum of $(n - 3N + 1)$ squares, namely the squares of the positive real numbers $\binom{n-3N}{i}\sqrt{(L(q_1 \oplus 1) - \epsilon)^i (L(1 \oplus q_2) - \epsilon)^{n-3N-i}}$. Therefore, setting $d = n - 3N + 1$, we can apply (3) to the sum on the right-hand side of (2). Using the Binomial Theorem, this leads to the inequality

$$\begin{aligned}
&\left(\sqrt{L(q_1 \oplus 1) - \epsilon} + \sqrt{L(1 \oplus q_2) - \epsilon}\right)^{2(n-3N)} \\
&= \sum_{i=0}^{n-3N} \binom{n-3N}{i} \sqrt{(L(q_1 \oplus 1) - \epsilon)^i (L(1 \oplus q_2) - \epsilon)^{n-3N-i}} \\
&\leq (n - 3N + 1) \sum_{i=0}^{n-3N} \binom{n-3N}{i}^2 (L(q_1 \oplus 1) - \epsilon)^i (L(1 \oplus q_2) - \epsilon)^{n-3N-i}.
\end{aligned}$$

Finally, comparing this with (2), we see that

$$S_n(q) \geq \frac{1}{n - 3N + 1}\left(\sqrt{L(q_1 \oplus 1) - \epsilon} + \sqrt{L(1 \oplus q_2) - \epsilon}\right)^{2(n-3N)}$$

for all $\epsilon > 0$. Taking nth roots, then taking limits as n goes to infinity, this implies that $L(q) \geq \left(\sqrt{L(q_1 \oplus 1)} + \sqrt{L(1 \oplus q_2)}\right)^2$ as claimed. □

A direct application of Theorem 4.2 provides the example for the high value of $L(q)$ that we promised in the introduction.

Example 4.3. Let $q_1 = 132$, and let $q_2 = 213$. Then Theorem 4.2 shows that

$$L(1324657) = 4L(1324) \geq 37.4.$$

Repeated applications of Theorem 4.2 lead to the following corollary.

Corollary 4.4. *Let $q_h = 1324657 \ldots (3h + 1)(3h + 3)(3h + 2)(3h + 4)$. That is, $q_h = 1324 \oplus 1324 \oplus \cdots \oplus 1324$. Then*

$$L(q_h) = h^2 L(1324).$$

A *layered pattern* is a pattern that consists of decreasing subsequences (the layers) so that the entries increase among the layers. For instance, 3217654 is a layered pattern. Let us call a layered pattern *almost increasing* if its layer lengths are all equal to 1 or 2, and layers of length 2 are never consecutive, and the first and last layers are of length 1. (This last requirement is not really necessary since we know [3] that $L(12 \oplus r) = L(21 \oplus r)$, but it will make discussion simpler.) Using Theorem 4.2, it is straightforward to compute $L(q)$ for any almost increasing layered pattern q, in terms of $L(1324)$.

Example 4.5. Let us compute $L(124356879)$. We get

$$\begin{aligned}
\sqrt{L(124356879)} &= \sqrt{L(12)} + \sqrt{L(13245768)} = \sqrt{L(12)} + \sqrt{L(13245)} + \sqrt{L(1324)} \\
&= \sqrt{L(12)} + \sqrt{L(1324)} + \sqrt{L(12)} + \sqrt{L(1324)} = 2 + \sqrt{L(1324)}.
\end{aligned}$$

We see that if q is an almost increasing layered pattern, then each sequence of consecutive layer lengths $(1, 2, 1)$ in q contributes $\sqrt{L(1324)} > 3$ to $L(q)$, while each sequence $(1, 1, 1, 1)$ contributes only 3.

Theorem 4.6. *Let q be an almost increasing layered pattern that has t sequences of consecutive layer lengths* $(1,\ 2,\ 1)$, *and u be the largest number so that u layers of length 1 can* simultaneously *be removed from q so that q is still an almost increasing layered pattern with t sequences of consecutive layer lengths* $(1,\ 2,\ 1)$. *Then*

$$L(q) = (t\sqrt{L(1324)} + u)^2.$$

The proof is straightforward by induction on the number of layers of length 1. One only needs to note that q can be "split" at any interior layer of length 1, and then Theorem 4.2 can be applied.

Therefore, the more $(1, 2, 1)$-sequences of consecutive layer lengths q has, the higher $L(q)$ is. This provides a way to immediately compare the Stanley–Wilf limits of two almost increasing layered patterns. This is the first result of this kind, that is, when such a comparison is possible on this large a family of patterns.

These results may be a first step in proving a long-standing conjecture that among all patterns of length k, the maximum of $L(q)$ is attained when q is a layered pattern, with layer lengths $(1, 2, 2, \ldots, 2)$ if k is odd, and $(1, 2, 2, \ldots, 2, 1)$, if k is even.

Theorem 4.6 can be strengthened a little bit by recalling a result of Backelin et al. [3] implying that for any pattern r, and any positive integer v, the equality $L(12\cdots v \oplus r) = L(v\cdots 21 \oplus r)$ holds. This shows that the reach of Theorem 4.6 could be extended to layered patterns whose first layer is of arbitrary length, and whose remaining part is an almost increasing layered pattern. Indeed, in patterns with that property, that first layer could be reversed, leading to an almost increasing layered pattern.

Finally, note that for most patterns q_1 and q_2, Theorem 4.2 has the following implication. We say most patterns, since it is easy to prove that as k goes to infinity, the ratio of indecomposable patterns among all patterns of length k converges to 1.

Corollary 4.7. *Let r_1 and r_2 be any two* indecomposable *patterns, and let $q = r_1 \oplus 1 \oplus r_2$. Then*

$$\sqrt{L(q)} \geq \sqrt{L(r_1)} + \sqrt{L(r_2)} + 2.$$

Proof. Let $q_1 = r_1 \oplus 1$, and let $q_2 = 1 \oplus r_2$. As both r_1 and r_2 are indecomposable, we know from a result in [7] that $\sqrt{L(q_1)} \geq \sqrt{L(r_1)} + 1$, and $\sqrt{L(q_2)} \geq \sqrt{L(r_2)} + 1$. The claim now follows from Theorem 4.2. $\square$

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Appendix

All that is left is to fulfill a promise made in the proof of Theorem 4.2 by proving the following Proposition.

Proposition A.1. *For any pattern q_1, we have*

$$L(q_1 \oplus 1) \geq L(q_1) + 1.$$

Note that if q_1 is indecomposable, then the much stronger statement that $L(q_1 \oplus 1) \geq (\sqrt{L(q_1)} + 1)^2$ is true as is proved in [7].

Proof (*of Proposition A.1*). Let us call an entry of a permutation a *right-to-left maximum* if it is larger than all entries on its right. Then the right-to-left maxima of a permutation always form a decreasing subsequence.

To construct a permutation p of length n that avoids $q_1 \oplus 1$ and has exactly M right-to-left maxima, choose any $M - 1$ positions out of the first $n - 1$ positions, and fill them with the entries $n, n - 1, \ldots, n - M + 2$ in decreasing order, then place the entry $n - M + 1$ into the last position of the permutation. Fill the remaining slots with any q_1-avoiding permutation p' of length $n - M$. The permutation p obtained has exactly M right-to-left maxima, namely the entries $n, n - 1, \ldots, n - M + 1$, and avoids $q_1 \oplus 1$. Indeed, if p contained a copy of $q_1 \oplus 1$, then any right-to-left maximum could only play the role of the last entry of q in that copy of q, but then p' would have to contain the rest of that copy, that is, a copy of q_1. That is impossible since p' was chosen to be q_1-avoiding.

This shows that

$$S_n(q_1 \oplus 1) \geq \sum_{M=1}^{n} \binom{n-1}{M-1} S_{n-M}(q_1). \tag{4}$$

Let $\epsilon > 0$, and let N be a constant such that $S_n(q_1) > (L(q_1) - \epsilon)^n$ for $n > N$. Then the previous displayed inequality yields that there exists a constant K such that

$$S_n(q_1 \oplus 1) \geq (L(q_1) - \epsilon + 1)^{n-1} - Kn^N,$$

since the few terms on the right-hand side of (4) for which $n - M \leq N$ are easy to bound from above. Taking nth roots, then taking limits as n goes to infinity, our claim is proved. □

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